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ASYMPTOTIC STRESS FIELD AROUND A CRACK NORMAL TO THE PLY-INTERFACE OF AN ANISOTROPIC COMPOSITE LAMINATE

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Abstract—Based upon the method of eigenfunction expansion and Lekhnitskii's complex-variable potentials for generalized plane strain deformations, the asymptotic stress field is examined for composite laminates with transverse cracks, including stress singularities and stress intensities. Due to the material symmetry, the decomposition of deformations into the rotationally symmetric and antisymmetric parts leads to reduced eigenvalue equations for the stress singularities. The boundary collocation using the asymptotic representations for the stress and displacement fields is shown to determine the stress throughout the cross-section of the composite laminates [90°/0°], and [0°/90°], when the transverse cracks are formed on the 90° ply under plane strain extension.

1. INTRODUCTION

Recently, the rapid increase in applications of advanced fiber composite laminates to many engineering structures and components has led to significant efforts in research on the mechanics of composite laminates. One of the important problems that has received increasing attention in the mechanics of composite laminates is how to deal with local deformation and complex stresses near geometric discontinuities and structural defects, such as edges, cutouts, cracks, and re-entrant corners, which are prevalent in almost all of the advanced composite materials and structures owing to fabrication and joining requirements as well as design considerations. Generally, the difficulties involve local stress singularities and inherently three-dimensional state of complex stresses. Moreover, the high local stresses and associated deformations caused by these structural and material discontinuities always result in undesirable delamination and transverse crack initiation and growth, leading to the final fracture. Thus, this class of mechanics problems has been of significant interest to researchers in mechanics of materials.

Among others, the stress singularities inherent to the material and geometric discontinuities of composite body composed of dissimilar isotropic materials were reported for the wedge problem by several authors; for example, Bogy (1971a) used the Mellin transform technique to treat the problem of wedges that are bonded together along a common edge, and Dempsey and Sinclair (1979) considered general wedge problems in composite bodies composed of many isotropic materials. For an anisotropic composite body, several authors examined the stress singularities and stress distributions near the free edge in composite laminates (see Wang and Choi, 1982; Zwiers *et al.*, 1982; Ting and Chou, 1981). The problems of the delamination cracks in an anisotropic composite laminate were treated, in the same way as the free edge problem, by Wang (1984) and Wang and Choi (1983). The stress singularity near the transverse cracks in anisotropic composite laminates was first reported by Ting and Hoang (1984) although solutions to similar problems were reported for isotropic bi-materials somewhat earlier (see, for example, Zak and Williams, 1963; Bogy, 1971b; Cook and Erdogan, 1972). In order to understand the fundamental failure behavior of anisotropic composite laminates with transverse cracks, we need detailed knowledge not only of the stress singularities but also of the stress distribution near the crack tip.

In this work, asymptotic stress behavior and stress redistribution near a transverse crack in an anisotropic composite laminate is studied based upon the elasticity theory. To simplify the problem, we assume that the laminate considered is under the generalized plane

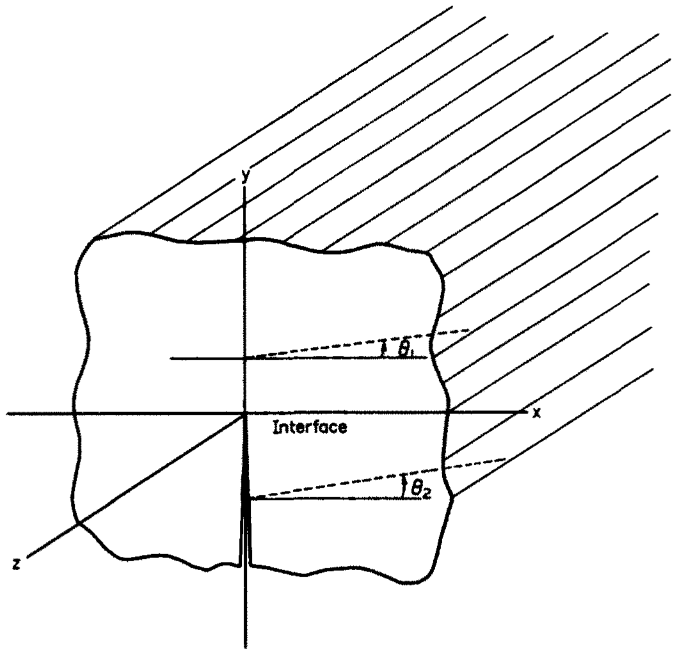

Fig. 1. Crack normal to the material interface.

strain deformation (see Lekhnitskii, 1963) and that the crack is normal to the material interface of the composite laminate and it terminates at the interface. The theoretical development is based on Lekhnitskii's complex-variable stress potentials (Lekhnitskii, 1963) and an asymptotic expansion method. The system of coupled governing partial differential equations is solved using the general solution in the form of complex-variable potential functions and its series expansion. General expressions for determining stress singularities, associated with local structural and material discontinuities, are shown in detail. The material symmetry is exploited to decompose the displacement field into the rotationally symmetric part and antisymmetric part, and this is shown to simplify the algebra involved in solving the eigenvalue problem associated with the stress singularity, compared with the work reported earlier by Ting and Hoang (1984). For a numerical example, the stress redistribution around the crack in a cross-ply laminate is investigated using the foregoing asymptotic representation in terms of eigenfunctions, with the aid of the boundary collocation method, which is a standard numerical technique for wedge problems. The stress intensity is computed and the stress distribution is plotted along the length of a laminate. This is followed by a brief discussion of the numerical result.

2. DEFORMATION OF AN ANISOTROPIC COMPOSITE BODY WITH A CRACK NORMAL TO AN INTERFACE

2.1. *Statement of the problem and basic equations*

Consider a composite body composed of two different anisotropic materials which are rigidly joined to each other along a straight interface. We assume that both of the materials have reflection–elastic symmetry about planes parallel to the interface, which is the case in fibrous composite materials. The body has a crack which extends perpendicularly to the interface and terminates right there, so that the tip of the crack meets the interface at right angles (Fig. 1). A rectangular Cartesian coordinate system is taken with its origin at the crack tip; the y axis is upward along the crack ligament, and the x axis is along the interface and perpendicular to the crack surface. The body is sufficiently long in the z direction that its geometry is assumed to be independent of the z coordinate. We consider a loading which does not vary along the z direction. Then the stresses become independent of the z coordinate, and the so-called generalized plane deformation is obtained (see

Lekhnitskii, 1963). By setting the axial strain along the z axis to be zero, this is further reduced to the generalized plane strain case, for which all three components of displacement are functions of x and y only.

Based upon the plane strain theory for an anisotropic body, we will examine the asymptotic solutions for the stress field around the transverse crack tip, including the stress singularities. This problem was treated by Ting and Hoang (1984) using Stroh's approach (Stroh, 1962). Although Stroh's approach would lead to a neater formulation, we use Lekhnitskii's complex potentials here (see Lekhnitskii, 1963), which may be regarded as the natural extension of the complex potentials for isotropic materials to the case of anisotropic materials. The material symmetry is exploited to reduce the algebra involved in calculating the eigenvalue problem associated with the stress singularities. That is, the deformation is decomposed into a rotationally symmetric part and a rotationally antisymmetric part, and use is made of the associated conditions along the crack ligament. This not only elucidates whether the deformation associated with each singularity or eigenvalue is rotationally symmetric or antisymmetric, but enables us to obtain the asymptotic solutions including stress singularities in an efficient way.

We regard the stress as a vector in the six-dimensional vector space, and use the notation σ_i for $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}, \sigma_{zx}, \sigma_{xy}$ whenever it is convenient. Similarly, the engineering strain components ε_i is defined in a Cartesian coordinate system by

$$\varepsilon_1 = \varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_2 = \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \varepsilon_3 = \varepsilon_{zz} = \frac{\partial w}{\partial z},$$

$$\varepsilon_4 = \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \varepsilon_5 = \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z},$$

$$\varepsilon_6 = \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \tag{1}$$

where u, v, w (the notation u_1, u_2, u_3 is to be used whenever convenient) are the components of displacement in the x, y, z directions, respectively. The stress–strain law for each of the two materials is then written as

$$\varepsilon_i = S_{ik}\sigma_k \quad (i,k = 1,2,3,\ldots,6),$$

where S_{ik} is the compliance matrix. Introducing the reduced compliance matrix defined by

$$\tilde{S}_{ik} = S_{ik} - S_{3i}S_{k3}/S_{33} \quad (i,k \neq 3), \tag{2}$$

we can rewrite the stress–strain law as

$$\varepsilon_i = \tilde{S}_{ik}\sigma_k \quad (i,k = 1,2,4,5,6). \tag{3}$$

For the generalized plane deformations, Lekhnitskii (1963) showed that the stress components are given in terms of the stress functions $F(x, y)$ and $\Psi(x, y)$ by

$$\sigma_{xx} = \frac{\partial^2 F}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 F}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 F}{\partial x\,\partial y}$$

$$\sigma_{xz} = \frac{\partial \Psi}{\partial y}, \quad \sigma_{yz} = -\frac{\partial \Psi}{\partial x}, \tag{4}$$

so that the equilibrium equations are satisfied identically. The stress component σ_{zz} and the displacement components u_i are given as

$$
\begin{aligned}
\sigma_{zz} &= (A_1x+A_2y+A_3)/S_{33}-S_{i3}\sigma_i/S_{33} \quad (i = 1,2,4,5,6)\\
u &= -A_1z^2/2-A_4yz+U(x,y)\\
v &= -A_2z^2/2+A_4xz+V(x,y)\\
w &= (A_1x+A_2y+A_3)z+W(x,y)
\end{aligned} \tag{5}
$$

where A_1 and A_2 are constants related to bending in the x–z and y–z planes, and A_3, A_4 represent the axial extension along the z axis and the twist about the z axis, respectively. Using the preceding eqns (1)–(4) and following Lekhnitskii (1963), we can show that $F(x,y)$ and $\Psi(x,y)$ satisfy the linear coupled partial differential equations,

$$
\begin{aligned}
L_3F+L_2\Psi &= -2A_4+A_1S_{34}/S_{33}-A_2S_{35}/S_{33}\\
L_4F+L_3\Psi &= 0,
\end{aligned} \tag{6}
$$

where L_2, L_3 and L_4 are linear partial differential operators of the second, third and fourth order, respectively, defined by

$$
\begin{aligned}
L_2 &= \tilde{S}_{44}\frac{\partial^2}{\partial x^2}-2\tilde{S}_{45}\frac{\partial^2}{\partial x\,\partial y}+\tilde{S}_{55}\frac{\partial^2}{\partial y^2}\\
L_3 &= -\tilde{S}_{24}\frac{\partial^3}{\partial x^3}+(\tilde{S}_{25}+\tilde{S}_{46})\frac{\partial^3}{\partial x^2\,\partial y}-(\tilde{S}_{14}+\tilde{S}_{56})\frac{\partial^3}{\partial x\,\partial y^2}+\tilde{S}_{15}\frac{\partial^3}{\partial y^3}\\
L_4 &= \tilde{S}_{22}\frac{\partial^4}{\partial x^4}-2\tilde{S}_{26}\frac{\partial^4}{\partial x^3\,\partial y}+(2\tilde{S}_{12}+\tilde{S}_{66})\frac{\partial^4}{\partial x^2\,\partial y^2}-2\tilde{S}_{16}\frac{\partial^4}{\partial x\,\partial y^3}+\tilde{S}_{11}\frac{\partial^4}{\partial y^4}.
\end{aligned} \tag{7}
$$

The solutions of the coupled partial differential eqns (6) in general consist of the homogeneous solution and the particular solution, see Lekhnitskii (1963), so that we write

$$
F = F^{h}+F^{p}, \quad \Psi = \Psi^{h}+\Psi^{p}, \tag{8}
$$

where the superscripts "h" and "p" indicate the homogeneous solution and the particular solution, respectively. The corresponding displacement field (5) can also be decomposed into

$$
\begin{aligned}
u &= u^{p}(x,y,z)+u^{h}(x,y)\\
v &= v^{p}(x,y,z)+v^{h}(x,y)\\
w &= w^{p}(x,y,z)+w^{h}(x,y).
\end{aligned} \tag{9}
$$

Note that the particular solution disappears when the loading parameters A_1, A_2, A_3, A_4 become zero, and the displacement field is a function of x and y only, so that the deformation is reduced to the generalized plane strain deformation ($\varepsilon_{zz} = 0$). In general, singular terms may arise not only in the homogeneous solutions but also in the particular solutions, and the particular solutions are sometimes given in simple polynomials depending upon the loading parameter A_i ($i = 1,2,3,4$) and the elastic stiffness tensor (Zwiers *et al.*, 1982). However, the dominant singular terms almost always occur in the homogeneous solution. Therefore, we hereafter restrict our attention to the generalized plane strain problem ($A_i = 0$, $i = 1,2,3,4$) corresponding to the homogeneous solution of the generalized plane deformations

2.2. *Interface continuity conditions and boundary conditions*

In this section we consider the conditions resulting from the continuity of tractions and displacements along the ply interface and the boundary conditions on the crack surface.

Together with the governing equations discussed in the previous section, these conditions will determine the structures of asymptotic stress field near the crack tip.

Assuming the two plies are perfectly bonded along the interface, which lies along the x axis (Fig. 1), we can impose the continuity of traction and displacement along the interface between the plies, $x < 0$ and $x > 0$ at $y = 0$,

$$\sigma_{yy}^{(1)} = \sigma_{yy}^{(2)}, \quad \sigma_{yz}^{(1)} = \sigma_{yz}^{(2)}, \quad \sigma_{xy}^{(1)} = \sigma_{xy}^{(2)}, \quad u^{(1)} = u^{(2)}, \quad v^{(1)} = v^{(2)}, \quad w^{(1)} = w^{(2)}, \tag{10}$$

where the superscripts (1), (2) refer to the upper and lower materials, respectively (see Fig. 1). The traction free conditions on the crack surfaces are given by

$$\sigma_{xx}^{(2)}(0^+, y) = 0, \quad \sigma_{xy}^{(2)}(0^+, y) = 0, \quad \sigma_{xz}^{(2)}(0^+, y) = 0 \tag{11}$$

$$\sigma_{xx}^{(2)}(0^-, y) = 0, \quad \sigma_{xy}^{(2)}(0^-, y) = 0, \quad \sigma_{xz}^{(2)}(0^-, y) = 0 \tag{12}$$

where 0^+ and 0^- indicate the right and left crack surfaces. Note that the stress and displacement fields depend only upon x and y because we limit ourselves to the plane strain solution (the homogeneous solution) as aforementioned. Because there is a discontinuity of an elastic field across a crack, we need to consider the left and the right sides of the lower portion separately in Fig. 1. Hence we have the three separate regions ($x > 0, y < 0$; $x < 0, y < 0$; $y > 0$) in each of which the governing equations (6) hold; 12 conditions (six along each of the positive and negative x axes) are then provided by eqn (10), and six conditions by eqns (11), (12), so that we have the total 18 conditions which determine the structures of the asymptotic homogeneous solutions. Ting and Hoang (1984) used these conditions based upon Stroh's approach to obtain the singularities at the crack tip. Here we decompose the deformations into the rotationally symmetric part and the rotationally antisymmetric part about the y axis and consider only the right half ($x > 0$) of the domain for each part of the deformations. It is noted that this decomposition is possible because the materials have reflection elastic symmetry about planes parallel to the x–z plane (see Appendix 1 for proof). For a deformation rotationally symmetric about the y axis,

$$u(x,y) = -u(-x,y), \quad v(x,y) = v(-x,y), \quad w(x,y) = -w(-x,y)$$

which can be rewritten for the upper ply as

$$\frac{\partial u^{(1)}(0,y)}{\partial y} = 0, \quad \frac{\partial v^{(1)}(0,y)}{\partial x} = 0, \quad \frac{\partial w^{(1)}(0,y)}{\partial y} = 0 \tag{13}$$

along the positive y axis. For a deformation rotationally antisymmetric about the y axis,

$$u(x,y) = u(-x,y), \quad v(x,y) = -v(-x,y), \quad w(x,y) = w(-x,y)$$

which also can be recast for the upper ply into

$$\frac{\partial u^{(1)}(0,y)}{\partial x} = 0, \quad \frac{\partial v^{(1)}(0,y)}{\partial y} = 0, \quad \frac{\partial w^{(1)}(0,y)}{\partial x} = 0 \tag{14}$$

along the positive y axis. Thus the six interface conditions (10) along the positive x axis, the three traction free conditions (11) on the right crack surface and conditions (13) determine the structures of the asymptotic solutions which are rotationally symmetric in u, v, w about the y axis. On the other hand, conditions (10), (11), (14) determine the structures of the asymptotic solutions which are rotationally antisymmetric in the displacements. A linear combination of these two will give the structure of solutions for general cases as obtained by Ting and Hoang (1984). This is due to the fact that the structure of the asymptotic eigenfunction solution and the associated eigenvalues are independent of the remote boundary conditions, and dependent only upon the aforementioned near field conditions. The present approach not only shows which deformation mode each term in the asymptotic solution is associated with but also makes computation involved in solution much easier, particularly the computation of the eigenvalues of higher order terms, as will be shown in the next section.

2.3. *Stress singularities and asymptotic solutions*

The homogeneous asymptotic solution of the coupled partial differential eqns (6) can be determined from the general structure of the solution and the near field conditions (10), (11), (13) and (14). The structure of the solution is determined from the characteristics of the elliptic partial differential eqns (6) on the complex plane. Equations (6) involve the sixth order, and their six characteristics, which appears as three pairs of complex conjugates, are given by (see Lekhnitskii, 1963)

$$z_k^{(\alpha)} = x + \mu_k^{(\alpha)} y, \quad z_{k+3}^{(\alpha)} = \bar{z}_k^{(\alpha)} \quad (k = 1, 2, 3). \tag{15}$$

Here $\alpha = 1, 2$ indicate the upper and lower materials and $\mu_k^{(\alpha)}$ are the roots of the algebraic characteristic equation

$$l_4^{(\alpha)}(\mu) l_2^{(\alpha)}(\mu) - l_3^{(\alpha)^2}(\mu) = 0$$

where

$$l_2^{(\alpha)}(\mu) = \tilde{S}_{55}^{(\alpha)}\mu^2 - 2\tilde{S}_{45}^{(\alpha)}\mu + \tilde{S}_{44}^{(\alpha)}$$
$$l_3^{(\alpha)}(\mu) = \tilde{S}_{15}^{(\alpha)}\mu^3 - (\tilde{S}_{14}^{(\alpha)} + \tilde{S}_{56}^{(\alpha)})\mu^2 + (\tilde{S}_{25}^{(\alpha)} + \tilde{S}_{46}^{(\alpha)})\mu - \tilde{S}_{24}^{(\alpha)}$$
$$l_4^{(\alpha)}(\mu) = \tilde{S}_{11}^{(\alpha)}\mu^4 - 2\tilde{S}_{16}^{(\alpha)}\mu^3 + (2\tilde{S}_{12}^{(\alpha)} + \tilde{S}_{66}^{(\alpha)})\mu^2 - 2\tilde{S}_{26}^{(\alpha)}\mu + \tilde{S}_{22}^{(\alpha)}.$$

For simplicity, we omit the superscript (α) hereafter in cases where there is no confusion. The homogeneous solutions for the stress functions $F(x, y)$, $\Psi(x, y)$, which depend upon z_k, can be found to have the forms (Lekhnitskii, 1963)

$$F(z_k) = \sum_{k=1}^{6} F_k(z_k), \quad \Psi(z_k) = \sum_{k=1}^{6} \eta_k F_k'(z_k) \tag{16}$$

where

$$\eta_k = -l_3(\mu_k)/l_2(\mu_k) = -l_4(\mu_k)/l_3(\mu_k).$$

We try a series expansion of a power type in z_k,

$$F_k(z_k) = \sum_{n=1}^{\infty} C_{kn} z_k^{\delta_n+2}/(\delta_n+1)(\delta_n+2) \tag{17}$$

where the eigenvalues δ_n are to be determined such that the near field conditions are satisfied. From (4), (16) and (17), we then find that

$$\sigma_i = \sum_{n=1}^{\infty} \sum_{k=1}^{3} [C_{kn}\Lambda_{ik} z_k^{\delta_n} + C_{(k+3)n}\bar{\Lambda}_{ik}\bar{z}_k^{\delta_n}] \quad (i \neq 3), \tag{18a}$$

where

$$\Lambda_{1k} = \mu_k^2, \quad \Lambda_{2k} = 1, \quad \Lambda_{4k} = -\eta_k, \quad \Lambda_{5k} = \mu_k \eta_k, \quad \Lambda_{6k} = -\mu_k. \tag{18b}$$

Using these and the stress–strain law (3), the strain–displacement relations (1) can be integrated to give the expressions for the displacements (see Lekhnitskii, 1963)

$$u_i = \sum_{n=1}^{\infty} \sum_{k=1}^{3} [C_{kn}\Gamma_{ik} z_k^{\delta_n+1} + C_{(k+3)n}\bar{\Gamma}_{ik}\bar{z}_k^{\delta_n+1}]/(\delta_n+1) \tag{19a}$$

where

$$\begin{aligned}
\Gamma_{1k} &= \tilde{S}_{11}\mu_k^2 + \tilde{S}_{12} - \eta_k \tilde{S}_{14} + \tilde{S}_{15}\eta_k\mu_k - \tilde{S}_{16}\mu_k \\
\Gamma_{2k} &= \tilde{S}_{12}\mu_k + \tilde{S}_{22}/\mu_k - \tilde{S}_{24}\eta_k/\mu_k + \tilde{S}_{25}\eta_k - \tilde{S}_{26} \\
\Gamma_{3k} &= \tilde{S}_{14}\mu_k + \tilde{S}_{24}/\mu_k - \tilde{S}_{44}\eta_k/\mu_k + \tilde{S}_{45}\eta_k - \tilde{S}_{46}.
\end{aligned} \tag{19b}$$

We emphasize that eqns (16) through (19) should be considered for each of the upper and lower materials separately.

Solutions (18) and (19) are required to satisfy the near field conditions. Thus substitution of (18) and (19) into (10), (11), (13) for the rotationally symmetric deformations and into (10), (11), (14) for the rotationally antisymmetric deformations gives two systems of homogeneous linear algebraic equations

$$\sum_{j=1}^{12} \Delta_{ij}^{(s)}(\delta_n) D_{jn}^{(s)} = 0, \quad \sum_{j=1}^{12} \Delta_{ij}^{(a)}(\delta_n) D_{jn}^{(a)} = 0, \quad (i = 1, 2, 3, \ldots, 12) \tag{20}$$

where $\Delta_{ij}^{(s)}$, $\Delta_{ij}^{(a)}$ are 12×12 matrices, $D_{jn} = C_{jn}^{(1)}$, $D_{(j+6)n} = C_{jn}^{(2)}$ $(j = 1 \sim 6)$, and the superscripts (s), (a) indicate the rotationally symmetric and antisymmetric deformations, respectively. For the existence of nontrivial solutions, we have

$$|\Delta_{ij}^{(s)}(\delta_n)| = 0, \quad |\Delta_{ij}^{(a)}(\delta_n)| = 0, \tag{21}$$

which determine the eigenvalues δ_n. Without decomposing the deformation, one will obtain, instead of (20) and (21), a set of 18×18 equations from the conditions (10), (11), (12) and the associated determinant equation, as given in Ting and Hoang (1984). In this case, the six eigenvectors C_{kn} for the lower left hand part $(x < 0,\ y < 0)$ are related to the six eigenvector for the lower right part $(x > 0,\ y < 0)$ either through rotational symmetry or through rotational antisymmetry with respect to the y axis (Fig. 1). Thus, the same solution as in the current approach can be obtained for the stress and displacement fields. It would however, involve more work to calculate the eigenvalues δ_n and eigenvectors C_{kn} from these 18×18 equations; particularly, numerical computation of δ_n with large absolute values from the determinant of the 18×18 matrix may be more difficult because of overflow. The present approach also elucidates whether the deformation mode associated with each eigenvalue is rotationally symmetric or antisymmetric.

For numerical examples we use the following two material data:

material data 1

$$\begin{aligned}
&E_L = 20 \times 10^6 \text{ psi}, \quad E_T = E_Z = 2.1 \times 10^6 \text{ psi} \\
&G_{LT} = G_{TZ} = G_{ZL} = 0.85 \times 10^6 \text{ psi} \\
&\nu_{LT} = \nu_{TZ} = \nu_{ZL} = 0.21,
\end{aligned} \tag{22a}$$

material data 2

$$\begin{aligned}
&E_L = 20 \times 10^6 \text{ psi}, \quad E_T = E_Z = 2.1 \times 10^6 \text{ psi} \\
&G_{LT} = G_{LZ} = 0.85 \times 10^6 \text{ psi}, \quad G_{TZ} = 0.6 G_{LZ} \\
&\nu_{LT} = \nu_{LZ} = 0.21, \quad \nu_{TZ} = 0.32,
\end{aligned} \tag{22b}$$

Table 1. Stress singularities for transverse cracks normal to the material interface of the composite laminate [0°/θ_2]

Material data	Deformation modes	θ_2 ($\theta_1 = 0$) 0°	15°	30°	45°	60°	75	90°
Material data 1	Rotationally symmetric	−0.5 −0.5	−0.5622 −0.4413	−0.6098 −0.3972	−0.6314 −0.3679	−0.6205 −0.3503	−0.5661 −0.3421	−0.5000 −0.3411
	Rotationally antisymmetric	−0.5	−0.4989	−0.4962	−0.4931	−0.4906	−0.4893	−0.4890
Material data 2	Rotationally symmetric	−0.5 −0.5	−0.5708 −0.4370	−0.6264 −0.3948	−0.6543 −0.3698	−0.6495 −0.3561	−0.6024 −0.3500	−0.5405 −0.3489
	Rotationally antisymmetric	−0.5	−0.4979	−0.4927	−0.4864	−0.4807	−0.4771	−0.4760

where L, T and Z indicate the fiber, transverse, and thickness directions of each ply, respectively. Both of these data sets are for graphite/epoxy, and the first was used by several authors, for example, Wang and Choi (1982) and, Ting and Hoang (1984). The second is, however, found to be more realistic. For ply orientation, we use the standard notation [θ_1/θ_2] where θ_α is the angle measured counterclockwise from the positive x axis to the fiber direction on the x–z plane (Fig. 1). Assuming that the crack is located at the layer of orientation θ_2, we consider [0°/θ_2] for various values of θ_2, and tabulate the result in Table 1. As seen from this table, two out of the three singularities represent the rotationally symmetric deformations, and the remaining one the rotationally antisymmetric deformation.

Because the strain energy is positive definite and the displacements at the crack tip are finite, the real part of δ_n should be greater than -1. From expression (18), we then find that the eigenvalues δ_n whose real part are between -1 and 0 characterize the order of stress singularity.

When the eigenvalues δ_n are known, within unknown constants the eigenvectors C_{jn} are computed from (20) and asymptotic homogeneous solutions for the stresses and displacements are given by (18) and (19). From the structure of $\Delta_{ij}(\delta_n)$, we can show that if δ_n is a root of the characteristic equation, so is its complex conjugate $\bar{\delta}_n$, and the expressions for the stresses and displacements become real. Using the superscript α (= 1, 2) to distinguish between the upper and lower materials again, we take

$$
\begin{aligned}
C_{kn}^{(\alpha)} &= \tfrac{1}{2}(\gamma_{1n} - i\gamma_{2n})b_{kn}^{(\alpha)} \quad \text{for complex } \delta_n, \operatorname{Im}[\delta_n] > 0, \\
C_{kn}^{(\alpha)} &= \tfrac{1}{2}\gamma_{3n} b_{kn}^{(\alpha)} \quad \text{for real } \delta_n
\end{aligned}
\tag{23}
$$

where $b_{kn}^{(\alpha)}$ is the solution for $C_{kn}^{(\alpha)}$, computed from one of eqns (20) by an appropriate normalization, and $\gamma_{1n} - i\gamma_{2n}$ and γ_{3n} are constants to be determined to complete the solution. The asymptotic expressions for the stresses and displacements (18), (19) are then written as

$$\sigma_i^{(\alpha)} = \sum_{n=1}^{\infty} P_{ni}^{(\alpha)}, \tag{24}$$

$$u_j^{(\alpha)} = \sum_{n=1}^{\infty} Q_{nj}^{(\alpha)}, \tag{25}$$

where $P_{ni}^{(\alpha)}$ and $Q_{nj}^{(\alpha)}$ are given by

$$P_{ni}^{(\alpha)} = \gamma_{1n} \operatorname{Re}\left\{\sum_{k=1}^{3} (b_{kn}^{(\alpha)}\Lambda_{ik}^{(\alpha)} z_k^{(\alpha)\delta_n} + b_{(k+3)n}^{(\alpha)}\bar{\Lambda}_{ik}^{(\alpha)} \bar{z}_k^{(\alpha)\delta_n})\right\} + \gamma_{2n} \operatorname{Im}\left\{\sum_{k=1}^{3} (b_{kn}^{(\alpha)}\Lambda_{ik}^{(\alpha)} z_k^{(\alpha)\delta_n} + b_{(k+3)n}^{(\alpha)}\bar{\Lambda}_{ik}^{(\alpha)} \bar{z}_k^{(\alpha)\delta_n})\right\}$$

if δ_n is complex, or

$$P_{ni}^{(\alpha)} = \gamma_{3n} \operatorname{Re}\left\{\sum_{k=1}^{3} b_{kn}^{(\alpha)}\Lambda_{ik}^{(\alpha)} z_k^{(\alpha)\delta_n}\right\} \quad \text{if } \delta_n \text{ is real,} \quad (i = 1,2,4,5,6)$$

and

$$Q_{nj}^{(\alpha)} = \gamma_{1n} \operatorname{Re}\left\{\sum_{k=1}^{3} (b_{kn}^{(\alpha)}\Gamma_{jk}^{(\alpha)} z_k^{(\alpha)\delta_n+1} + b_{(k+3)n}^{(\alpha)}\bar{\Gamma}_{jk} \bar{z}_k^{\delta_n+1})/(\delta_n+1)\right\} + \gamma_{2n} \operatorname{Im}\left\{\sum_{k=1}^{3} (b_{kn}^{(\alpha)}\Gamma_{jk}^{(\alpha)} z_k^{(\alpha)\delta_n+1} + b_{(k+3)n}^{(\alpha)}\bar{\Gamma}_{jk}^{(\alpha)} \bar{z}_k^{(\alpha)\delta_n+1})/(\delta_n+1)\right\}$$

if δ_n is complex, or

$$Q_{nj}^{(\alpha)} = \gamma_{3n} \operatorname{Re}\left\{\sum_{k=1}^{3} b_{kn}^{(\alpha)}\Gamma_{jk}^{(\alpha)} z_k^{(\alpha)\delta_n+1}/(\delta_n+1)\right\} \quad \text{if } \delta_n \text{ is real,} \quad (j = 1,2,3).$$

Here only one of the complex conjugates is included when δ_n is complex. Because of this and because summation for k is taken only up to 3 when δ_n is real, the factor $\frac{1}{2}$ in (23) does not appear in (24) and (25). When the region is symmetric about the y axis and the loading is rotationally symmetric or antisymmetric, only the associated eigenvalues should be included in (24), (25), and it is sufficient to consider only the right half ($x > 0$) of the entire region. For general cases, both the symmetric and antisymmetric eigenvalues should be included, and the whole region (both $x > 0$ and $x < 0$) needs to be considered. In those cases, expressions (24) and (25) for the lower material ($\alpha = 2$) are valid only for $x \geqslant 0$ because of the discontinuity across the crack. But the expressions for the region $x < 0$ can be obtained from the symmetry and antisymmetry; for a rotationally symmetric deformation, v, σ_{xx}, σ_{yy}, σ_{xz} become even functions of x and u, σ_{yz}, σ_{xy} are odd functions, while for the rotationally antisymmetric deformation, u, w, σ_{yz}, σ_{xy} are even, and v, σ_{xx}, σ_{yy}, σ_{xz} are odd. To obtain the expressions for the stress and displacement field for the region $y < 0$ and $x < 0$, on the right-hand sides of (24), (25), for $\alpha = 2$ we thus replace x by $-x$ for v, σ_{xx}, σ_{yy}, σ_{xz}, and x by $-x$, and γ_{in} by $-\gamma_{in}$ for u, w, σ_{yz}, σ_{xy} if δ_n is associated with the rotationally symmetric mode; if δ_n is associated with the rotationally antisymmetric deformation, x is replaced by $-x$ for all the components and then γ_{in} are replaced by $-\gamma_{in}$ only for v, σ_{xx}, σ_{yy}, σ_{xz}. Hence the asymptotic expressions for the region $x < 0$ of the lower material are obtained for general loadings and geometry.

The unknown constants γ_{in} can be determined by matching the asymptotic expressions with far field conditions (the conditions at the remote boundary) through the boundary collocation technique (Wang and Choi, 1982), or through the hybrid finite element method in which the present asymptotic solutions are incorporated into a singular crack tip element (Wang and Yuan, 1982).

In composite wedge problems, it is well known that the power-type expansion (17) may break down for some wedge angles and elastic constants (see Dempsey and Sinclair, 1979; Ting and Chou, 1981 and the references cited therein). This occurs when the eigenvalues δ_n are multiple roots and there are not enough sets of the power-type eigenvectors associated with these multiple eigenvalues. Dempsey and Sinclair (1979) resolved this difficulty by introducing logarithmic eigenfunctions, which ensure the existence of the sets of eigenvectors enough to span the solution. Subsequently this was extended to the problem of anisotropic composite laminates by Ting and Chou (1981). The existence of the logarithmic eigenfunctions can be examined by calculating the multiplicity of the eigenvalues and the rank of the associated coefficient matrices $\Delta_{ij}^{(s)}(\delta_n)$, $\Delta_{ij}^{(a)}(\delta_n)$ in (20).

2.4. *Orthotropic materials*

For orthotropic materials like cross-ply laminates, the two stress functions $F(x, y)$ and $\Psi(x, y)$ are decoupled from each other. The stress function $F(x, y)$ is related to the stress components σ_{xx}, σ_{yy}, σ_{xy} and the displacement components u, v. On the other hand, $\Psi(x, y)$ is related to the components σ_{xz}, σ_{yz} and w. Here we limit our attention to the stress function $F(x, y)$ which describes the plane strain deformations when the axial displacement component w is identically zero. All the loading parameters A_i $(i = 1, 2, 3, 4)$ disappear identically under the plane strain deformations, and the governing equation for $F(x, y)$ can be written as

$$L_4 F(x, y) = 0, \tag{26}$$

where L_4 is given by eqn (7) (with $\tilde{S}_{26} = \tilde{S}_{16} = 0$ due to the material symmetry). For the boundary conditions, we impose the same conditions as in Section 2.2 for the traction components related to σ_{xx}, σ_{xy}, and the displacement components u, v.

The solution of the partial differential eqn (26) has the general form

$$F(x, y) = \sum_{k=1}^{4} F_k(x + \mu_k y), \tag{27a}$$

where μ_k are the roots of the following algebraic equation,

$$l_4(\mu) = \tilde{S}_{11}\mu^4 + (2\tilde{S}_{12} + \tilde{S}_{66})\mu^2 + \tilde{S}_{22} = 0. \tag{27b}$$

Introducing the power-type expansion (17) and imposing the aforementioned boundary conditions, we can obtain a set of 8 by 8 linear homogeneous equations and the associated eigenvalue problem similar to eqns (21), for each of the rotationally symmetric and antisymmetric deformations,

$$\sum_{\beta=1}^{8} \Delta_{\alpha\beta}^{(s)}(\delta_n) D_{\beta n}^{(s)} = 0, \quad \sum_{\beta=1}^{8} \Delta_{\alpha\beta}^{(a)}(\delta_n) D_{\beta n}^{(a)} = 0 \quad (\alpha = 1 \sim 8)$$

$$|\Delta_{\alpha\beta}^{(s)}(\delta_n)| = 0, \quad |\Delta_{\alpha\beta}^{(a)}(\delta_n)| = 0 \tag{28}$$

where $D_{\beta n} = C_{\beta n}^{(1)}$, and $D_{(\beta+4)n} = C_{\beta n}^{(2)}$ $(\beta = 1 \sim 4)$.

Once the eigenvalues δ_n are determined, the relationship among $C_{pn}^{(m)}$ $(m = 1, 2, p = 1 \sim 4)$ can be found from eqn (28). Splitting the complex free constants $C_{pn}^{(m)}$ into real and imaginary parts as in eqns (23), we can find expressions for the stress components σ_1, σ_2, σ_6 $(= \sigma_{xx}, \sigma_{yy}, \sigma_{xy})$ and the displacement components u_1 and u_2 $(= u, v)$ in terms of the real free constants γ_{1n}, γ_{2n} and γ_{3n}. These expressions are obtained by taking $i = 1, 2, 6$ in eqn (24) and $j = 1, 2$ in eqn (25), and by summing each term for $k = 1$ and $k = 2$ only.

In the preceding development, we have limited ourselves to the case of orthotropic materials, for which the characteristic roots μ_k are distinct. For isotropic materials, the operator L_4 in eqn (26) is reduced to the biharmonic operator and the characteristic eqn (27b) has double roots, $\mu_k = \pm i$. For double roots, it can be shown that $\mathrm{d}F_k(z_k)/\mathrm{d}\mu_k = \mathrm{d}f_k(z_k, \mu_k)/\mathrm{d}\mu_k$ is another solution for each double root, where we have written $F_k(z_k) = f_k(z_k, \mu_k)$ to emphasize that $F_k(z_k)$ is dependent upon μ_k not only through z_k but also in other ways. When both μ_k are double roots, eqn (27a) then takes the form (see Lekhnitskii, 1963)

$$F(x,y) = \sum_{k=1}^{2}\left[F_k(Z_k)+\frac{\mathrm{d}F_k(z_k)}{\mathrm{d}\mu_k}\right] = \sum_{k=1}^{2}\left[f_k(z_k,\mu_k)+\frac{\mathrm{d}f_k(z_k,\mu_k)}{\mathrm{d}\mu_k}\right], \tag{29}$$

which comprises four terms again. A brief manipulation, as given in Appendix 2, shows that this is reduced to Goursat's formula,

$$F(x,y) = \bar{z}\varphi(z)+\chi(z)+z\overline{\varphi(z)}+\overline{\chi(z)},$$

where $\varphi(z)$ and $\chi(z)$ are Muskhelishvili's complex potentials (Muskhelishvili, 1963). Accordingly for a body composed of two dissimilar *isotropic* materials, the present approach, with the general solution (29), is reduced to the approach by Zak and Williams (1963), who used the eigenfunction expansion for the product solutions of the stress function $F(x,y)$, which can be obtained from Goursat's formula above. This class of problems for two dissimilar *isotropic* materials was also successfully treated using the Mellin transform technique, for example, Bogy (1971b) and Cook and Erdogan (1972).

As a concluding remark for this section, we add that transverse damage cracks occurring under external loading run parallel to the fiber orientation (see Crossman and Wang, 1982), so that the present development, based upon the generalized plane strain assumption, is not applicable to this class of problems unless the fiber orientation of a cracked ply is perpendicular to the stretching direction or the x axis (which is the case for cross-ply laminates loaded in the direction of principal material coordinates). For angle ply laminates the present result may, however, be applicable to composite laminates under extension with initial cracks running perpendicular to the stretching direction (the cracks may not be parallel to the fiber orientation).

3. NUMERICAL EXAMPLES: EXTENSION OF A CROSSPLY LAMINATE WITH TRANSVERSE CRACKS

In this section, we use the boundary collocation method in conjunction with the asymptotic representation (24) and (25) to find the stress field near transverse cracks in cross-ply laminates. Because the transverse cracks occur running parallel to the fiber orientation of 90° ply, and the loading is perpendicular to this fiber orientation, the preceding development is applicable to this case. We assume that the laminate dimension in the z direction is sufficiently large compared with the laminate thickness so that the composite laminate is in the state of the plane strain deformations on the x–y plane. After the cracks occur under increasing loads in the laminates, the stress field is perturbed near the crack tip and it can be represented by eqn (24). The free constant γ_{in} in this equation is to be determined in such a way that the general solution matches the remote boundary conditions approximately through collocation. We consider the extension of cross-ply laminates $[90°/0°]_s$ and $[0°/90°]_s$ that have transverse cracks in 90° ply. We assume that the cracks have a uniform spacing, and the arrangement of cracks is thus obtained by repetition of the unit cell as shown in Fig. 2. Because of the geometric symmetry, we then have only to consider the upper right part ($0 \leqslant x \leqslant b$, $-h \leqslant y \leqslant h$) of the unit cell, where h and b are the ply thickness and half the crack spacing, respectively. For material data, we use (22b), which are more realistic compared with (22a). The remote boundary conditions include the traction free conditions on the top surface ($y = h$, $0 \leqslant x \leqslant b$) and the symmetry conditions at $x = b$ and on the middle plane, so that

$$\sigma_{yy}(x,h) = 0, \qquad \sigma_{xy}(x,h) = 0 \tag{30}$$

$$u(b,y)-\bar{u} = 0, \qquad \frac{\partial v(b,y)}{\partial x} = 0 \tag{31}$$

$$\frac{\partial u(x,-h)}{\partial y} = 0, \quad \frac{\partial v(x,-h)}{\partial x} = 0, \tag{32}$$

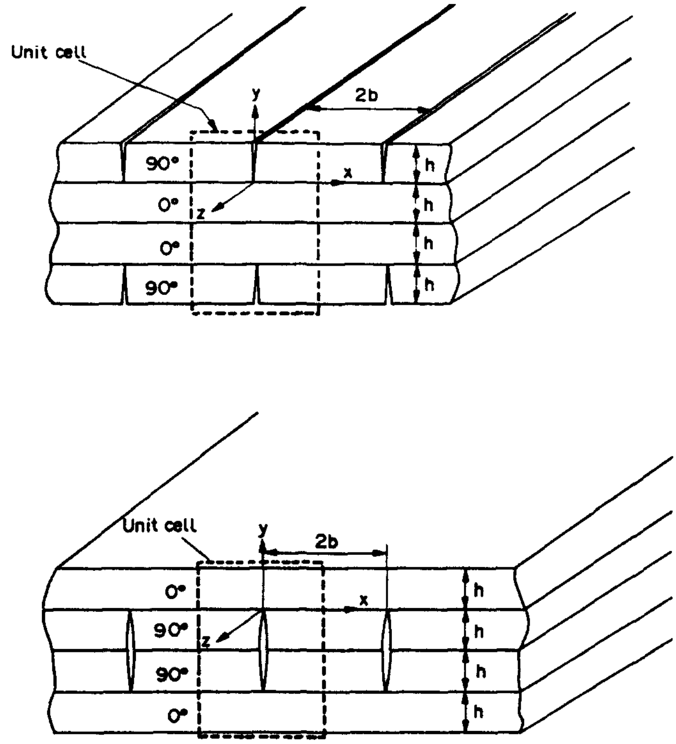

Fig. 2. Transverse cracks in $[90°/0°]_s$ and $[0°/90°]_s$.

where $\bar{u}$ is the displacement at $x = b$ relative to $x = 0$, thus representing the extension of the half crack spacing. The free constants γ_{in} in eqns (24) and (25) are now determined so as to result in the least square residual of the above conditions along the boundary by using the boundary collocation method (see Appendix 3 for details).

For transverse cracks, the stress does not have the square root singularity at the crack tip, as seen in Table 1, and therefore the stress intensity factor as in the linear elastic fracture mechanics of homogeneous materials does not exist in the strict sense. However, we may define the stress intensity to be an asymptotic magnification factor of stress along a given direction at the crack tip. Important directions are along the negative y axis (self-similar direction in homogeneous materials) and along the x-axis (material interface). We therefore define

$$K_I = \lim_{r \to 0} r^{-\delta_s} \sigma_{xx}^{(m)}(r, \phi) = \gamma_{31} \, \mathrm{Re}\left[\sum_{k=1}^{3} b_{k1}^{(m)} \mu_k^{(m)^2} e^{i\phi\delta_s} \right]$$

$$(m = 2, \phi = -\pi/2 \quad \text{for } [90°/0°]_s, \quad \text{and} \quad m = 1, \phi = \pi/2 \quad \text{for } [0°/90°]_s)$$

$$K_i = \lim_{r \to 0} r^{-\delta_s} \sigma_i^{(m)}(r, 0) = \gamma_{31} \, \mathrm{Re}\left[\sum_{k=1}^{3} b_{k1}^{(m)} \Lambda_{ik}^{(m)} \right] \quad (i = 1, 2, 3, 6)$$

where $\delta_s (= \delta_1 = -0.3489)$ is the real eigenvalue associated with the singularity [the other singular eigenvalue $\delta_s = -0.5405$ is associated with the stress function $\Psi(x, y)$ or with the stress components σ_{xz}, σ_{yz}, and therefore it is not active under the extension].

The intensity factor K_I represents the amplitude of the singular stress field for the opening mode along the direction of the crack tip. On the other hand, $K_2^{(1)} = K_2^{(2)}$, $K_6^{(1)} = K_6^{(2)}$ among K_i, and these two denote the amplitude of the singular normal stress σ_{yy} and shear stress σ_{xy} along the interface, respectively, and therefore these parameters are related to the delamination tendency. The computed results for K_I and K_i are tabulated in

Table 2. Convergence of K_I versus the numbers of eigenvalues and collocation stations in $[90°/0°]_s$ with $b/h = 8$

No. of eigenvalues	No. of collocation stations	$(K_I/(\bar{u}\ b)) \times 10^{-6}$
31	56	3.9157
	76	3.9163
	96	3.9163
	116	3.9163
38	56	3.9190
	76	3.9188
	96	3.9198
	116	3.9198
46	56	3.9381
	76	3.9413
	96	3.9393
	116	3.9394
54	56	3.9473
	76	3.9490
	96	3.9494
	116	3.9488

Tables 2 and 3. Table 2 shows the convergence of the K_I values versus the number of eigenvalues and the number of collocation stations. We see that the present boundary collocation method has a very stable convergence characteristic. In fact, the difference in the K_I values from those obtained through the singular hybrid finite element technique turns out to be negligibly small—the two agree with each other up to the first three digits (Im and Kim, 1989). Table 3 shows that the normal stress amplitude along the material interface is positive in both of the laminates $[0°/90°]_s$ and $[90°/0°]_s$. This signifies the possibility of delamination as the load increases. The shear stress field will also contribute to possible delamination. The stress distribution throughout the laminate is computed from eqn (24) by summing each term for $k = 1$ and 2 only. The numerical results are plotted for $[0°/90°]_s$ in Figs. 3–5 for $h/b = 8$. Figures 3 and 4 show the distributions of the axial and shear stress components σ_{xx}, σ_{xy} along the x axis, and the through-thickness stress distribution of the axial stress is plotted in Fig. 5. These figures reveal the nature of the significant perturbation of the stress field due to the presence of transverse cracks. The perturbed stress field reaches a uniform distribution around $x = 0.4b \sim 0.7b$ (or $x = 3.2h \sim 5.6h$), which corresponds to the classical laminate solution. The interlaminar normal stress component, although it is singular and characterized by K_2 near the crack tip, dies out more rapidly compared with the shear stress component σ_{xy}. It is negligibly small except for the crack tip region and therefore not plotted. At the center of crack ($x/b = 0$, $y/h = -1.0$), the stress component σ_{yy} is compressive and its magnitude is close to the nominal axial stress (σ_{xx} near $x/b = 1.0$) due to the effect of Poisson's ratio (see Fig. 3), as found in the Griffith crack under tension.

Table 3. Asymptotic stress amplitude along the material interface

Ply orientation		σ_{xx}	σ_{yy}	σ_{zz}	σ_{yz}	σ_{xz}	σ_{xy}
		$b/h=8$; $K_i/(\bar{u}/b) \times 10^{-6}$					
$[90°/0°]_s$	90°	0.9149	0.8645	0.3737	0.0	0.0	0.4238
	0°	6.269	0.8645	0.4149	0.0	0.0	0.4238
$[0°/90°]_s$	0°	5.863	0.8086	0.3881	0.0	0.0	−0.3964
	90°	0.8557	0.8086	0.3495	0.0	0.0	−0.3964

$$\left(K_i = \lim_{r \to 0} r^{-\delta_s} \sigma_i(r, 0), \quad \delta_s = -0.3489\right)$$

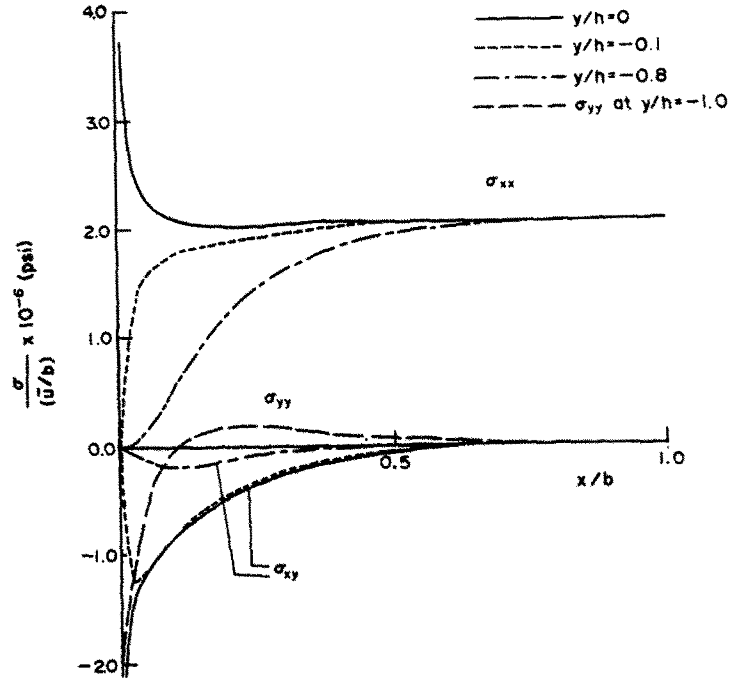

Fig. 3. Stress distribution along 90° ply in $[0°/90°]_s$.

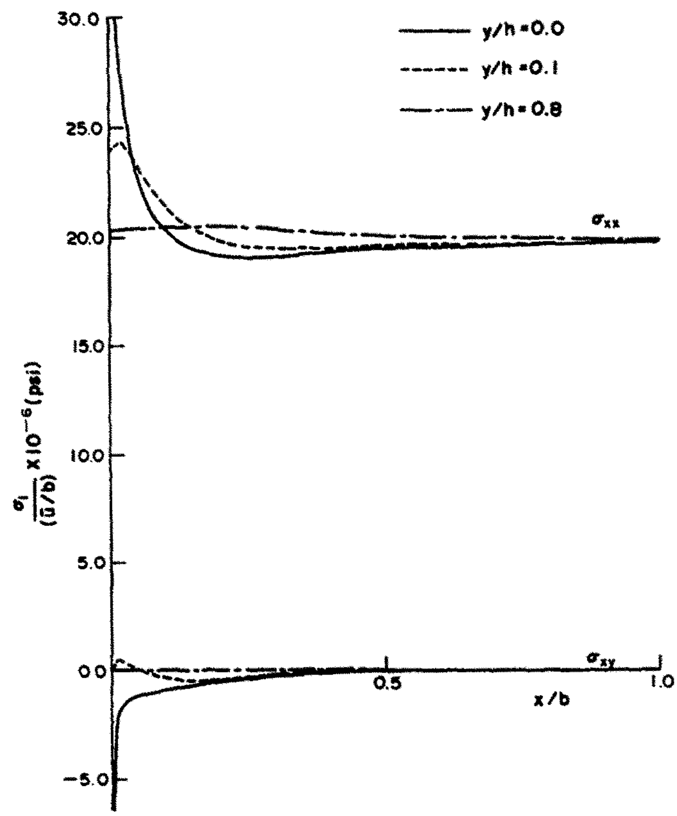

Fig. 4. Stress distribution along 0° ply in $[0°/90°]_s$.

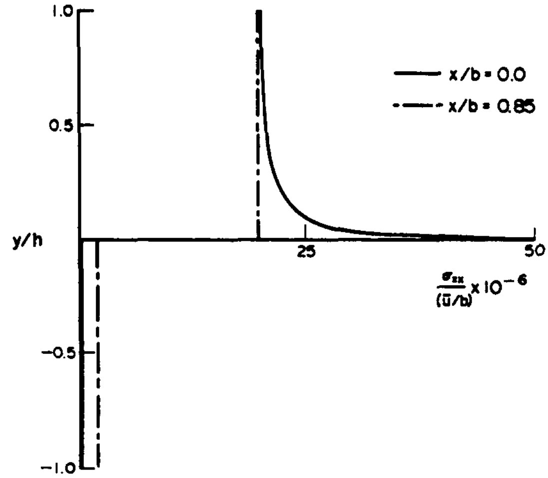

Fig. 5. Through-thickness distribution of the axial stress in $[0°/90°]_s$.

In general, the transverse cracks occurring in the 90° ply are firmly arrested at the interface and cannot penetrate into the 0° ply because the latter is much tougher. However, they grow along the interface as the load increases, so that delamination cracks are ultimately formed (Lim, 1988). The behavior of such cracks including their stability, which is important for the understanding of failure in cross-ply laminates, is under investigation at the KAIST, and will be reported later.

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APPENDIX 1

In Figs. 1 and 2, every plane parallel to the x–z plane is the plane of reflection elastic-symmetry for each ply of fiber-reinforced composite laminates as long as the fibers are parallel to the x–z plane. (This means that the 180° rotation about the y axis also belongs to a symmetry group.) We then have for the fourth-order stiffness tensor.

$$C_{1123} = C_{2223} = C_{3323} = C_{2313} = C_{1112} = C_{2212} = C_{3312} = C_{1312} = 0 \tag{A1}$$

and we are left with only 13 components.

Suppose we have a displacement field u, v, w which is under a generalized plane strain deformation [$\varepsilon_{zz} = 0$, but $w = w(x, y) \neq 0$]. We decompose the displacements into

$$u(x,y) = u^s(x,y)+u^a(x,y), \quad v(x,y) = v^s(x,y)+v^a(x,y), \quad w(x,y) = w^s(x,y)+w^a(x,y). \tag{A2}$$

Here the superscripts "s" and "a" denote the rotationally symmetric part and the antisymmetric part with respect to the y axis, each of which satisfies the relations

$$\begin{aligned} &u^s(x,y) = -u^s(-x,y), \quad v^s(x,y) = v^s(-x,y), \quad w^s(x,y) = -w^s(-x,y) \\ &u^a(x,y) = u^a(-x,y), \quad v^a(x,y) = -v^a(-x,y), \quad w^a(x,y) = w^a(-x,y). \end{aligned}$$

This leads to the relations

$$\begin{aligned} &\varepsilon^s_{xx}(x,y) = \varepsilon^s_{xx}(-x,y), \quad \varepsilon^s_{yy}(x,y) = \varepsilon^s_{yy}(-x,y), \quad \varepsilon^s_{zz}(x,y) = 0 \\ &\varepsilon^s_{yz}(x,y) = -\varepsilon^s_{yz}(-x,y), \quad \varepsilon^s_{xz}(x,y) = \varepsilon^s_{xz}(-x,y), \quad \varepsilon^s_{xy}(x,y) = -\varepsilon^s_{xy}(-x,y) \end{aligned} \tag{A3}$$

and

$$\begin{aligned} &\varepsilon^a_{xx}(x,y) = -\varepsilon^a_{xx}(-x,y), \quad \varepsilon^a_{yy}(x,y) = -\varepsilon^a_{yy}(-x,y), \quad \varepsilon^a_{zz}(x,y) = 0 \\ &\varepsilon^a_{yz}(x,y) = \varepsilon^a_{yz}(-x,y), \quad \varepsilon^a_{xz}(x,y) = -\varepsilon^a_{xz}(-x,y), \quad \varepsilon^a_{xy}(x,y) = \varepsilon^a_{xy}(-x,y). \end{aligned} \tag{A4}$$

From these relations and (A1), it follows that each of the stress fields σ^s_{ij} and σ^a_{ij} satisfies the same relations as ε^s_{ij} and ε^a_{ij}, respectively. Thus the stress and strains fields for the rotationally symmetric part and the antisymmetric part are decoupled from each other and the decomposition (A2) is valid.

APPENDIX 2

When all μ_k are double roots, it follows from eqn (29) that Lekhnitskii's general solution for $F(x, y)$ in Section 2.4 takes the form

$$F(x,y) = \sum_{k=1}^{2}\left[f_k(z_k,\mu_k)+\frac{\partial f_k(z_k,\mu_k)}{\partial \mu_k}+\frac{\partial f_k(z_k,\mu_k)}{\partial z_k}\frac{\mathrm{d}z_k}{\mathrm{d}\mu_k}\right].$$

Noting that

$$\frac{\mathrm{d}z_k}{\mathrm{d}\mu_k} = (z_k-\bar{z}_k)/(\mu_k-\bar{\mu}_k),$$

we rewrite the above equation as

$$\begin{aligned} F(x,y) &= \sum_{k=1}^{2}\left[f_k(z_k,\mu_k)+\frac{\partial f_k(z_k,\mu_k)}{\partial \mu_k}+\frac{z_k}{\mu_k-\bar{\mu}_k}\frac{\partial f_k(z_k,\mu_k)}{\partial z_k}-\frac{\bar{z}_k}{\mu_k-\bar{\mu}_k}\frac{\partial f_k(z_k,\mu_k)}{\partial z_k}\right] \\ &= \sum_{k=1}^{2}[\chi_k(z_k)+\bar{z}_k\varphi_k(z_k)] \end{aligned} \tag{A5}$$

where $\chi_k(z_k)$ represents the first three terms in the preceding expression, and $\bar{z}_k\varphi_k(z_k)$ the last term.

Because $\mu_1 = i$, $\mu_2 = -i$, $z_1 = z$ and $z_2 = \bar{z}$ for isotropic materials, we can take

$$\chi_1(z_1) = \chi(z), \quad \chi_2(z_2) = \bar{\chi}(\bar{z}) = \overline{\chi(z)}$$
$$\varphi_1(z_1) = \varphi(z), \quad \varphi_2(z_2) = \bar{\varphi}(\bar{z}) = \overline{\varphi(z)}$$

so that $F(x, y)$ may become real. Equation (A5) is then reduced to Goursat's formula.

APPENDIX 3

The free constants γ_{in} in eqns (24) and (25) are determined through boundary collocation so that the asymptotic expressions (24) and (25) meet, in the least square sense, the remote boundary conditions at the far field, such as eqns (29), (30) and (32). For convenience, eqns (24) and (25) are rewritten as

$$\sigma_i = \sum_{n=1}^{\infty} \beta_n g_i^n(x, y, \delta_n)$$
$$u_i = \sum_{n=1}^{\infty} \beta_n h_i^n(x, y, \delta_n).$$

Here β_n denote one of $\gamma_{in}(i = 1, 2, 3)$, and g_i^n, h_i^n represent the corresponding real or imaginary part in eqns (24) and (25). Let the prescribed values on the remote boundaries be $\bar{\sigma}_i$, $\bar{u}_i$, $\partial\bar{u}_i/\partial x$, $\partial\bar{u}_i/\partial y$ (i = 1, 2, 3). The boundary conditions can then be written as

$$\bar{\sigma}_i = \sum_{n=1}^{\infty} \beta_n g_i^n(\bar{x}, \bar{y}, \delta_n)$$
$$\bar{u}_i = \sum_{n=1}^{\infty} \beta_n h_i^n(\bar{x}, \bar{y}, \delta_n)$$
$$\frac{\partial \bar{u}_i}{\partial x} = \sum_{n=1}^{\infty} \beta_n h_{i,x}^n(\bar{x}, \bar{y}, \delta_n)$$
$$\frac{\partial \bar{u}_i}{\partial y} = \sum_{n=1}^{\infty} \beta_n h_{i,y}^n(\bar{x}, \bar{y}, \delta_n)$$

where $\bar{x}$, $\bar{y}$ denote the dimensionless coordinates of the remote boundaries under consideration, and $h_{i,x}^n$ and $h_{i,y}^n$ are the partial derivatives of h_i^n with respect to x and y, respectively.

The functional to be minimized in the boundary collocation may be written as

$$\pi = \int_{\partial B_R} \left[\sum_{i=1}^{6} A_i \left(\bar{\sigma}_i - \sum_{n=1}^{\infty} \beta_n g_i^n \right)^2 + \sum_{i=1}^{3} B_i \left(\bar{u}_i - \sum_{n=1}^{\infty} \beta_n h_i^n \right)^2 + \sum_{i=1}^{3} C_i \left(\frac{\partial \bar{u}_i}{\partial x} - \sum_{n=1}^{\infty} \beta_n h_{i,x}^n \right)^2 + \sum_{i=1}^{3} D_i \left(\frac{\partial \bar{u}_i}{\partial y} - \sum_{n=1}^{\infty} \beta_n h_{i,y}^n \right)^2 \right] \mathrm{d}s$$

where ∂B_R denotes the whole remote boundaries, and A_i, B_i, C_i and D_i are equal to 1 if the terms corresponds to the prescribed boundary conditions, and otherwise become zero. Minimization of the above residual leads to the following system of simultaneous linear equations for the free constants $\beta_n (= \gamma_{1n}, \gamma_{2n}, \gamma_{3n})$

$$\sum_{n=1}^{\infty} \beta_n \int_{\partial B_R} \sum_{i=1}^{3} [A_i g_i^m g_i^n + A_{i+3} g_{i+3}^m g_{i+3}^n + B_i h_i^m h_i^n + C_i h_{i,x}^m h_{i,x}^n + D_i h_{i,y}^m h_{i,y}^n] \, \mathrm{d}s$$
$$= \int_{\partial B_R} \sum_{i=1}^{3} \left[A_i \bar{\sigma}_i g_i^m + A_{i+3} \bar{\sigma}_{i+3} g_{i+3}^m + B_i \bar{u}_i h_i^m + C_i \frac{\partial \bar{u}_i}{\partial x} h_{i,x}^m + D_i \frac{\partial \bar{u}_i}{\partial y} h_{i,y}^m \right] \mathrm{d}s \quad (m = 1, 2, 3, 4, 5, \ldots).$$

The system of the above infinite number of linear equations can be approximated by an appropriate truncation, say $m, n = 1 \sim N$.